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SOME RESULTS CONCERNING APPROXIMATION OF REGULARIZED COMPRESSIBLE FLOW

HUNG DINH AND GRAHAM F. CAREY

Department of Aerospace Engineering and Engineering mechanics, University of Texas at Austin, Austin TX 78712–1085, U.S.A.

SUMMARY

We summarize here some theoretical results for fictitious gas regularization of compressible flow and give error estimates for the finite element approximation to the regularized problem.

KEY WORDS Compressible Fictitious Gas Finite Elements Error Estimates

INTRODUCTION

The non-linear full potential equation for compressible aerodynamics is elliptic if the flow is everywhere subsonic. However, when the flow accelerates near an aerofoil the velocity may be locally supersonic and the equation is of mixed type. An interesting observation resulting from a classical Rayleigh–Janzen perturbation analysis is that the first-order compressibility correction is independent of the ratio of specific heats γ for the gas. The choice $\gamma = -1$ was used by Chaplygin[1] and later by von Karman[2] in their analytical studies to simplify the compressible flow problem and yet still obtain useful results.

This idea of fictitious gas regularization has reappeared recently in relation to finite difference and finite element methods for computing compressible flows.[3,4] In the present study we examine the mathematical structure of the non-linear problem and give sufficient conditions on γ for existence of a unique solution. Error estimates for a finite element approximation are stated and verified in a numerical experiment. Details of the proofs are given elsewhere.[5]

FORMULATION

Let ρ be the density and $\mathbf{q}$ the velocity with $\mathbf{q} = \nabla\phi$ for potential ϕ. From conservation of mass

$$\nabla \cdot (\rho \mathbf{q}) = 0 \tag{1}$$

where $\rho = \rho(q)$. The adiabatic equation of state $p = k\rho^{\gamma}$ for the gas, in the equation of motion (Bernoulli relation) yields

$$\rho(q) = \left(1 - \frac{\gamma - 1}{2} q^2\right)^{1/(\gamma - 1)} \tag{2}$$

where $q = |\mathbf{q}|$ and γ is the ratio of specific heats for the gas.

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Substituting (2) in (1) with $\mathbf{q} = \nabla\phi$ we obtain the full potential equation. The flow is subsonic and the equation is elliptic if $q < a^* = [1 + (\gamma - 1)/2]^{-1/2}$ and hyperbolic if $q > a^*$. It is convenient to use the local Mach number

$$M^2 = \frac{-q}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}q} = \frac{q^2}{1 - [(\gamma - 1)/2]q^2} \tag{3}$$

so that $M < 1$ and $M > 1$ in the subsonic and supersonic regions, respectively.

For $\gamma = -1$ we obtain the Chaplygin gas approximation and (3) becomes

$$\rho(q) = (1 + q^2)^{-1/2} = (1 - M^2)^{1/2} \tag{4}$$

where $M^2 = q^2/(1 + q^2)$. Hence $M < 1$ for all q with this fictitious gas and $M \to 1$ as $q \to \infty$.

Now, the variational functional

$$J(v) = \int_\Omega \left[1 - \left(\frac{\gamma - 1}{2}\right)(\nabla v)^2\right]^{\gamma/\gamma - 1} \mathrm{d}x \tag{5}$$

on domain Ω has the full potential equation as its Euler equation, and we can investigate the properties of J to obtain appropriate conditions on γ for existence of a unique solution. In particular we find that the functional J in (5) is well defined on the Sobolev space $W^{1,p}(\Omega)$ if $\gamma > 1$ or $\gamma \leqslant -1$, and in the usual Hilbert space $H^1 = W^{1,2}$ if $\gamma \leqslant -1$ only. Further, J is strictly convex for a fictitious gas satisfying $\gamma \leqslant -1$. It follows that there exists a unique solution to the regularized problem with $\gamma \leqslant -1$.

For a finite element discretization Ω_h of Ω and approximation space $H^h \subset H^1(\Omega)$, convexity of J follows as in the continuous problem above. This property implies existence of a unique solution to the approximate problem.[6]

The finite element approximation satisfies the essential boundary conditions and the variational condition

$$\int_{\Omega_h} \rho_h \nabla\phi_h \cdot \nabla v_h \mathrm{d}x = 0, \qquad \text{for all } v_h \in H^h \tag{6}$$

where $\rho_h = \rho(q_h)$ for the fictitious gas relation ($\gamma \leqslant -1$). The following estimates for the error $\phi - \phi_h$ can be established:[5]

Lemma

For $\phi \in W^{1,\infty}(\Omega_h) \cap H^r(\Omega_h)$, $r \geqslant k + 1$ and C^0 elements of degree $k = 1$ and 2 there exists a constant C such that

$$\|\nabla\phi_h\|_{\infty,\Omega_h} \leqslant C \tag{7}$$

Theorem

For $\phi \in W^{1,\infty}(\Omega_h) \cap H^r(\Omega_h)$, $r \geqslant k + 1$ and $\|\nabla\phi_h\|_{\infty,\Omega_h} \leqslant C$, then we obtain the estimate

$$|\phi - \phi_h|_{1,\Omega_h} \leqslant Ch^k \tag{8}$$

NUMERICAL EXPERIMENT

To confirm rate in (8) we constructed a model non-linear problem with $\gamma = -2$. The data f

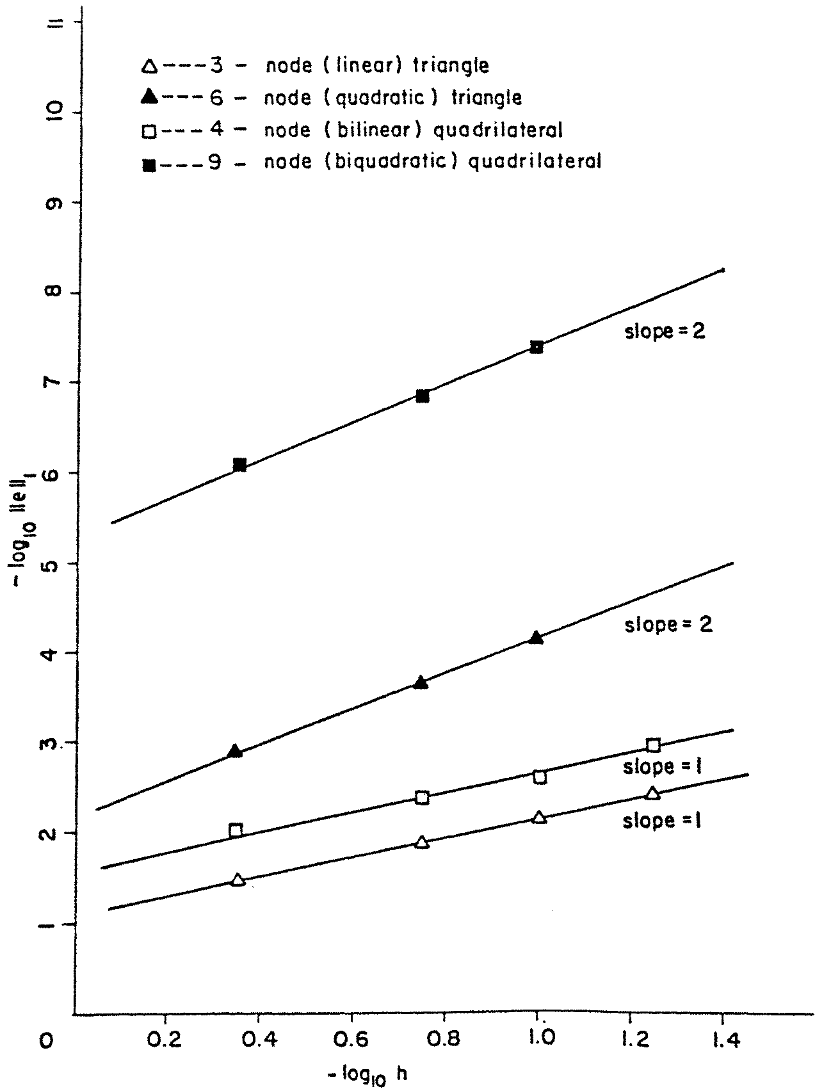

Figure 1. Optimal rate of convergence for the velocity potential in the H^1-norm

in $\nabla \cdot (\rho \nabla \phi) = f$ was constructed to correspond to the solution $\phi = 2x(y-1)\ \sin[y(x-1)]$ on $[0, 1] \times [0, 1]$. Plots of the rate of convergence in the H^1 norm are given in Figure 1 for calculations with linear and quadratic elements.

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